Self-anamorphic images

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Abstract

A curve is called self-anamorphic if it is the same shape as its reflection in a curved mirror except for rotation and rescaling. We show here that self-anamorphic curves exist for images seen in conical mirrors viewed from above. This is perhaps surprising because reflections seen in cones are typically so deformed that they have been used in the past to reveal images concealed in anamorphic art. Fourier analysis is used to find a general solution for self-anamorphic curves and four examples are illustrated. One of them is the familiar heart shape. Its unexpected appearance where it seems not to belong is reminiscent of the unexpected appearance of lifelike forms in the style of design known to art historians as the Grotesque.

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1.0 Geometry and the Grotesque in art

Images that span the boundary between mathematics and nature can approach that boundary from the side of nature or the side of mathematics. Whilst one can find many examples of objects, especially in architecture and art, that are geometrical it is rare to find purely geometrical objects that have a meaning outside of mathematics; when they do arise they can seem uncanny. For an example of what is possible in this direction consider coloring the following region consisting of all points *(x,y)* that satisfy

$$\sin[x(\cos y - \cos x)] + \sin x - \sin y > 0, \text{ for } -18 < x < -4 \text{ and } 4 < y < 18. \qquad (1)$$

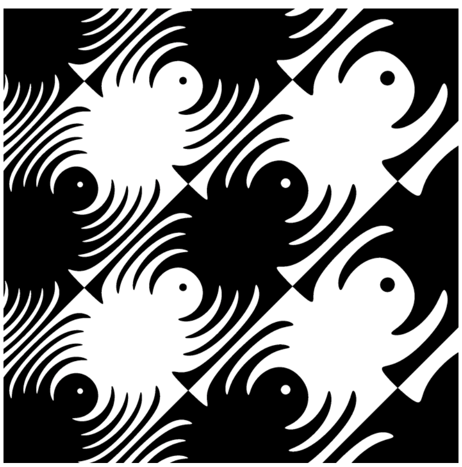

Figure 1. Shown in black is the region defined by equation (1). This example is due to Gary Tupper of Pedagoguery Software [1].

The result, shown in Figure 1, is unexpected because the equation that generates it seems too simple; in short it looks as if the bugs that appear have no right to be there. Of course this is only because it is easy to overlook how contrived the equation that generates them really is. In art the appearance of life where it is unexpected, in a way that challenges reason and order, is something associated with the Grotesque. Art historians use this word as a technical term [2, p.7], to refer to a type of decorative ornament fashionable between 1500 and 1800 that is nowadays not often studied but which has a long and subversive history going back to the time of the gargoyle and before. Its fanciful designs were often collaged compositions of vaguely organic shapes resembling bones or leaves. Such images possess a kind of ambiguity that allows them to be perceived either as inanimate designs or as something alive; in the twilight world of the Grotesque there is movement, purpose and life where it ought not exist. To take one example from many, in the first half of the seventeenth century the printmaker Bracelli created lively figures whose component parts were polygons, leaves, clouds, or objects from his studio [2, pp.7-9]. His work was admired by the Surrealists and through them the influence of the Grotesque has continued into our own time. Figure 1 shows that mathematically generated images have potential to be grotesque in this way. This occurs when they look like recognizable objects but the resemblance has no ready explanation. When that happens we prefer to project meaning and purpose on to the object itself rather than explain it away as an effect of chance.

Anamorphic drawings have many affinities with Grotesque designs. Both are ambiguous since they can both be read as orderly or chaotic depending upon the point of view. Although nowadays reduced to the status of a scientific toy, anamorphic art has in the past been an object of wonder. Its subject matter in its eighteenth century heyday; caricatures, erotic vignettes, and scenes of sorcery for a confidential public, represented material similar to that found in Grotesque art [3, pp.156-7]. If the mirror is seen as a way of crossing a boundary beyond which these grotesque images are kept safe, one is tempted to ask: are there any shapes that are unaffected by the mirror? The question is interesting because finding shapes that could cross such a boundary unscathed is a mathematical problem.

2.0 Geometry of anamorphic images

In drawing anamorphic art one is reminded of John Ruskin's advice to drawing students; "Never, by choice, draw anything polished" [4, p.108]. Ruskin was trying to protect the beginner's confidence by directing them away from subjects that are simply too difficult. To draw the world reflected in a spherical mirror such as Escher did in his *Hand with Reflecting Sphere*, 1935 [5, p.60], is very challenging, but it is even harder to draw something by eye so that its reflection in a shiny object is correct. It is because the obvious way to create an anamorphic drawing is so bewitchingly difficult that anamorphic art can seem impressive. Of course there are easier ways of proceeding, for a mirror with a simple shape the relationship between an image and its reflection will be a mapping controlled by a fairly simple function. If this is made explicit as a relation between a regular and a distorted grid then the drawing can be reduced to a mechanical procedure that is today quickly done by computer.

A survey of the different ways of presenting anamorphic drawings may be found in Hunt *et al* [6]. Let us go through them looking for the possibility of self-anamorphic images. We note, in passing, that images distorted in perspective so that they are seen correctly when viewed at a shallow angle, like traffic signs painted on roads, or Holbein's skull [7, p.13] cannot contain an invariant form. For a plane mirror one has only to choose an image with a line of mirror symmetry such as a face. We believe, but have not proved, that the stretched and bent images seen in cylindrical mirrors viewed at an angle cannot be self-anamorphic. If these trite or impossible cases are set aside then of the mirrors with simple shapes routinely used in anamorphic art only the cone remains. Looking down on a conical mirror is arguably the best way of presenting anamorphic drawings because, compared with cylindrical mirrors, the image is so strongly deformed that one is less likely to guess what the result is going to be before the mirror is put in place.

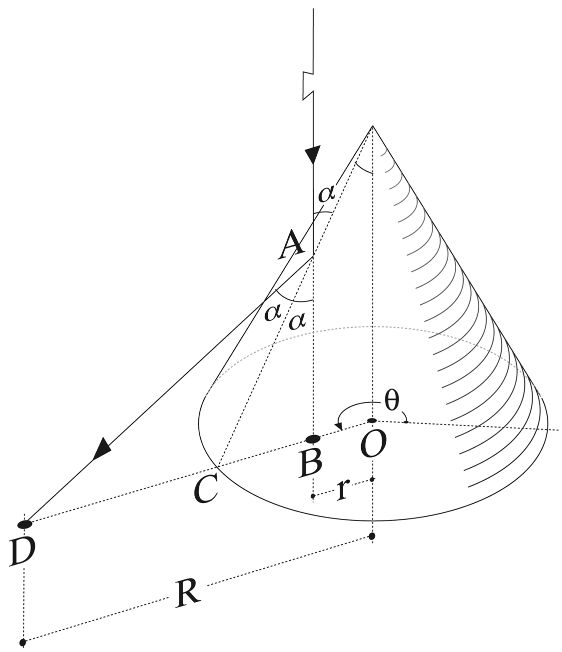

Figure 2: An image formed by a conical mirror with half-angle α.

Figure 3: A distorted image rectified by a 30˚ half-angle conical mirror.

Expressed mathematically the transformation wrought by a conical mirror is surprisingly simple. Figure 2 shows the geometrical optics for a right cone standing on an image that appears correct when its reflection in the cone is viewed vertically with the eye at infinity. Let the cone have half-angle α and unit radius, OC, at its base. Placing the mirror so that its axis lies over the origin O, let (R,θ) be the polar coordinates of a point D, on the (distorted) image, and (r,θ) be the coordinates of B, its reflection seen in the cone. Triangles ABC and ABD show us that

$$AB = \frac{1-r}{\tan\alpha} = \frac{R-r}{\tan 2\alpha}. \tag{2}$$

This gives us the relation between R and r. If $R = R(\theta)$ is a curve around the cone and $r = r(\theta)$ is its reflection in the cone then setting $\frac{\tan 2\alpha}{\tan\alpha} = \lambda > 2$,

$$r(\theta) = \gamma - \beta R(\theta). \tag{3}$$

Where: $$\gamma = \frac{\lambda}{(\lambda-1)}, \quad \beta = \frac{1}{(\lambda-1)}. \tag{4}$$

For an image to be visible $1 < R(\theta) < \lambda$. The lower limit prevents the curve going underneath the base of the cone where it cannot be seen and the upper limit stops it going beyond the event horizon at radius λ. When an object crosses this horizon it seems to vanish into the apex of the cone. Equation 3 can be used to write a computer program for transforming pixel images; the results for a photograph before and after reflection in a 30° half-angle cone are shown in Figure 3. Can anything survive such distortion unchanged? Note that when a closed curve is drawn around the cone all points outside that curve are reflected to points inside the reflected curve and the image is turned inside out. This means that there is a possibility of finding an edge that retains its shape under the transformation even though it may perhaps be rotated and rescaled about the origin. We will call such a curve self-anamorphic.

3.0 Self-anamorphic curves in conical mirrors

Using equation (3) a curve is self-anamorphic curve if

$$r(\theta) = \gamma - \beta R(\theta) = \mu R(\theta + \phi), \tag{5}$$

where the scaling factor, μ is less than 1, and the image is rotated about point O through an angle ϕ. Because $R(\theta) = R(\theta + 2n\pi)$, $n = 1,2,..$ the image is periodic with period 2π and can be expressed as a Fourier series

$$R(\theta) = \sum_{n=0,1,2,..} (a_n \cos n\theta + b_n \sin n\theta). \tag{6}$$

Substitution of (6) into (5) gives the condition for a self-anamorphic image

$$\gamma - \beta \sum_{n=0,1,2,..} a_n \cos n\theta + b_n \sin n\theta = \mu \sum_{n=0,1,2,..} a_n \cos n(\theta + \phi) + b_n \sin n(\theta + \phi). \tag{7}$$

When $n = 0$ we find that $a_0 = \dfrac{\gamma}{(\mu + \beta)}$. Subtracting this term from both sides leaves

$$\begin{aligned} &-\beta \sum_{n=1,2,..} \left(a_n \cos n\theta + b_n \sin n\theta \right) = \\ &\mu \sum_{n=1,2,..} \left(a_n (\cos n\theta \cos n\phi - \sin n\theta \sin n\phi) + b_n (\sin n\theta \cos n\phi + \cos n\theta \sin n\phi) \right) \end{aligned} \tag{8}$$

Equating the coefficients firstly of $\cos \mathrm{n}\theta$ then of $\sin n\theta$ gives

$$-\beta a_n = \mu (a_n \cos n\phi + b_n \sin n\phi) \tag{9}$$

$$-\beta b_n = \mu (-a_n \sin n\phi + b_n \cos n\phi). \tag{10}$$

Equations (9) and (10) may be expressed in matrix format as

$$\begin{pmatrix} \frac{\beta}{\mu} + \cos n\phi & \sin n\phi \\ -\sin n\phi & \frac{\beta}{\mu} + \cos n\phi \end{pmatrix} \begin{pmatrix} a_n \\ b_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{11}$$

The simultaneous equations in *a* and *b*, (9) and (10), have a non-trivial solution if and only if the determinant of the coefficient matrix in (11) is zero, in other words if

$$(\frac{\beta}{\mu} + \cos n\phi)^2 + \sin^2 n\phi = \left(\frac{\beta}{\mu}\right)^2 + 1 + \frac{2\beta}{\mu}\cos n\phi = 0. \tag{12}$$

The only solution of this is $\mu = \beta$ and $\cos n\phi = -1$. (13)

Substituting $\mu = \beta$ into equation (8), we find that $a_0 = \frac{\lambda}{2} = \frac{\tan 2\alpha}{2\tan\alpha}$. (14)

This completes the analysis. The general solution for a self-anamorphic curve appearing rotated through $\phi = \pi / m$ radians when seen in a cone with a half-angle α and unit radius at its base is

$$R(\theta) = \frac{\tan 2\alpha}{2\tan\alpha} + \sum_{n=m(2p+1),\ p=0,1,2,3,..} \left(a_n \cos n\theta + b_n \sin n\theta\right). \tag{15}$$

In summary: if the reflection of a self-anamorphic curve is rotated through an angle ϕ then provided that nϕ is an odd integer multiple of π, and provided that the curve stays within the limits $1 < R(\theta) < \lambda$, then the *a* and *b* coefficients in equation (15) can take any values. Note that there is no solution for $\phi = 0$, so aside from the trivial circle there is no unrotated self-anamorphic curve. Some allowable values of n for particular angles ϕ are as follows:

For images rotated by 180˚	n = 1, 3, 5, 7, …
For images rotated by 90˚	n = 2, 6, 10, 14, …

For images rotated by 60˚ $\quad n = 3, 9, 15, 21, \ldots$

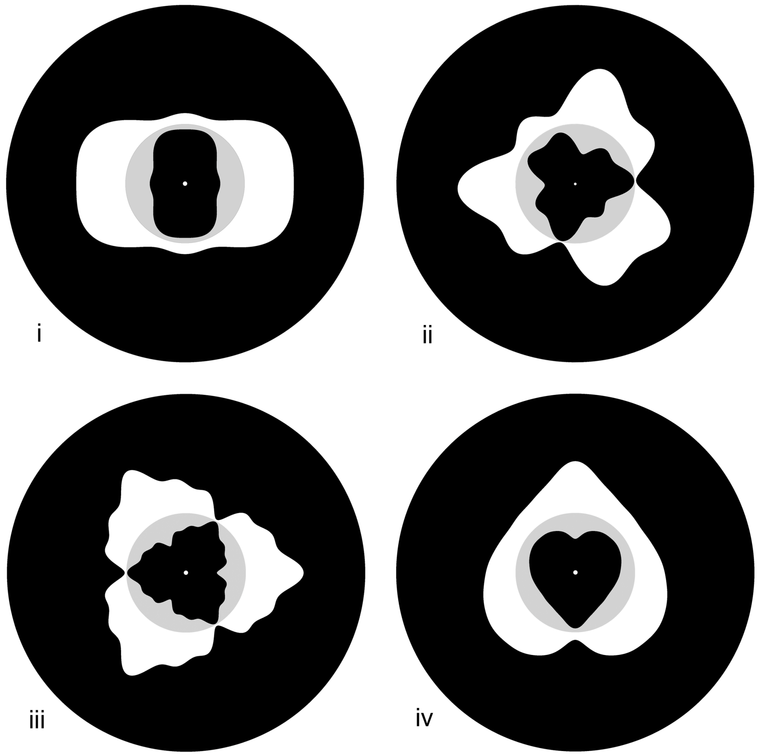

Figure 4: Examples of self-anamorphic images

Equation (15) can be used to draw self-anamorphic curves. Restricting ourselves to the first five or six terms of the series, Figure 4 shows four particular solutions for a 30˚ half-angle cone for which $\lambda = 1.5$. The equations that generate the outermost black regions bounded by a circle radius 3 are listed below. The grey circle with radius 1 represents the cone and the white spot is its apex. The reflections within the grey circles were generated using equation (3), which for this cone is $\mathrm{r}(\theta) = 3 - 2\mathrm{R}(\theta)$. Note that for clarity of presentation $(\theta - \frac{\pi}{2})$ has been used instead of θ in solution (iv) so that the inner heart appears upright in Figure 4.

(i) $3 > R > 1.5 + 0.4 \cos 2\theta - 0.08 \cos 6\theta$

(ii) $3 > R > 1.5 + 0.28\cos 3\theta + 0.14\sin 3\theta + 0.19\cos 7\theta$

(iii) $3 > R > 1.5 + 0.3\cos 3\theta + 0.1\cos 9\theta + 0.07\cos 15\theta$

(iv) $3 > R > 1.5 + 0.2\cos 3(\theta-\pi/2) + 0.072\cos 5(\theta-\pi/2) + 0.04\cos 7(\theta-\pi/2) + 0.028\cos 9(\theta-\pi/2) + 0.012\cos 11(\theta-\pi/2) + 0.008\cos 13(\theta-\pi/2)$

In solution (i) the cone rotates the image by 90˚ with respect to the object, in solution (iii) by 120˚, and in (ii) and (iv) by 180˚. The heart shape in solution, (iv), is rather striking. It might be possible to improve it by the addition of smaller terms but how many more would have to be added to make its point sharp is unknown to us. We conjecture that it is possible to achieve this sharpness because we were able to draw a self-anamorphic heart with a good point by eye. Figure 5 shows a photograph of our drawing reflected in a two inch steel cone of half-angle 20˚ taken with the camera about two feet above the cone. The image of the reflected heart was fairly robust, which is to say it could be seen clearly with the eye off-axis and close to the cone.

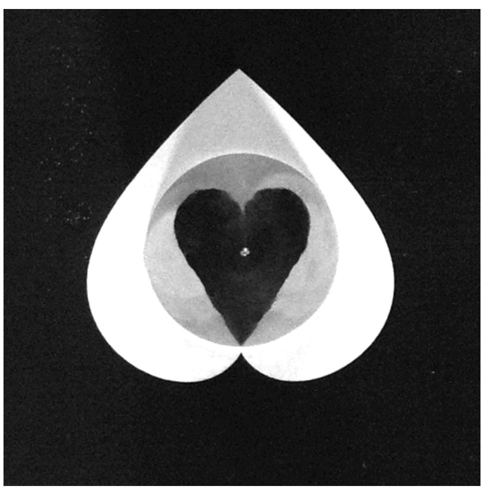

Figure 5: A photograph of a self-anamorphic heart seen in a 20˚ chromed steel cone.

4.0 Conclusion

The fact that a mirror can reproduce a rotated object is unexpected and rather charming. The appearance of the heart inverted in colour and orientation seems uncanny because it is so easy to invest it with meaning; inner love; the indestructibility of love; world turned around: secret love, all come to mind. As evidence of unexpected life-like form it is a rare example of a mathematical Grotesque. It is all an illusion of course, without smoke truly, but with a mirror certainly. Is the heart the only recognisable self-anamorphic image? One hesitates to say that it is, only that no other is known at present. Aside from the heart the best realistic shape we have been able to draw is shown in Figure 4 (ii) which a shape something between a bird and an angel. Improving on our efforts is left as a challenge to future investigators.

Acknowledgements

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References

[1] Gary Tupper G., 2003, personal communication.

[2] Griffiths A, 1995, The Grotesque, Ornamental Prints from the British Museum. The South Bank Centre, London.

[3] Connelly F, 2003, *Modern Art and the Grotesque*. C.U.P. Cambridge.

[4] Ruskin J, 1857, 1971, *The Elements of Drawing*, Dover Publications, N.Y.

[5]Escher M.C, Locher J.L, 1974, *The World of M.C. Escher*, Abrams, N.Y.

[6] Hunt J, Nickel B, & Gigault C, 2000, Anamorphic images. *Am. J. Physics*. 68 (3) pp. 232-237.

[7] Leeman F, 1976, *Hidden Images*, Abrams, N.Y.